WHITEPAPER: ACHIEVING THE CRAMER-RAO LOWER BOUND IN GPS TIME-OF-ARRIVAL ESTIMATION, A FREQUENCY DOMAIN WEIGHTED LEAST-SQUARES ESTIMATOR APPROACH

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1. Introduction

Recall, that for a coherent code phase detector, the variance of the code phase estimate error Δt, in seconds^2 can be derived as [1]

$$\sigma^2_{\Delta t} \approx \frac{\Delta_{\text{EL}} B_{\text{DLL}} T_C^2}{2\frac{C}{N_0}} \tag{1}$$

where

Δ_{EL} is the early-late correlator spacing, in chips,
B_{DLL} is the bandwidth of the delay-locked loop (DLL), in Hz, and
T_C is the GNSS signal PN-code chip duration, in seconds
$\frac{C}{N_0}$ is the carrier-to-noise ratio of the received signal.

It becomes natural to ask if the estimate of Δt using a delay locked loop is the best that one could do, i.e. achieves the smallest $\sigma^2_{\Delta t}$. The so-called Cramer-Rao lower bound (CRLB) provides us with a lower bound on $\sigma^2_{\Delta t}$. Any estimation strategy that achieves the CRLB is considered "efficient."

One simple estimation strategy that is known to achieve the CRLB is a batch weighted least squares estimator. By "batch" we mean that the estimator processes an entire chuck of data at once.

2. Batch Least Squares Estimator

We can define the cross correlation function as

$$R[\hat{t}_s] = E_C\left[\{C[\tau_j - t_s] + n(j)\}C[\tau_j - \hat{t}_s]\right] \tag{2}$$

$$\approx \frac{1}{N_k}\sum_{j=j_k}^{j_k+N_k-1}\{C[\tau_j - t_s] + n(j)\}C[\tau_j - \hat{t}_s] \tag{3}$$

$$\tag{4}$$

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where

E_C is the expectation over the GNSS PN-code
C is the PN-code sequence
n is the front-end white-Guassian noise
N_k is the number of samples over which the average will be performed
$\hat{t}_s$ is the code-phase estimate, in seconds
t_s is the true code-phase, in seconds

One simple way to estimate the true code phase offset t_s is to maximize $R[\hat{t}_s]$ over $\hat{t}_s$.

We will assume that we have accurate models for satellite and receiver motion, clocks, and atmospheric effects such that t_s remains constant throughout the entire batch interval (no dependence on τ_j. This is not typically the case (hence the need for a PLL and DLL), but it simplifies our derivation. It is also assumed that the signal has been shifted to baseband (i.e. no Doppler frequency remaining). The maximizing value of $\hat{t}_s$ is given only at the resolution of the signal sampling interval. To overcome this, we can transform the problem into a more convenient representation using the Wiener-Khinchin theorem. But, first lets define the code phase error $\Delta t = t_s - \hat{t}_s$ and define a new cross-correlation function in terms of this new term. This provides us with a more-familiar notion to us of the error between our *a priori* "guess" $\hat{t}_s$ and t_s.

$$
\begin{aligned}
R[\Delta t] = R[\hat{t}_s] &= E_C[\{C[\tau_j - t_s] + n(j)\}C[\tau_j - \hat{t}_s]] && (5)\\
&= E_C[\{C[\tau_j - t_s + t_s] + n(j)\}C[\tau_j - \hat{t}_s + t_s]] && (6)\\
&= E_C[\{C[\tau_j] + n(j)\}C[\tau_j - \underbrace{(t_s - \hat{t}_s)}_{\Delta t}]] && (7)\\
&= E_C[C[\tau_j]C[\tau_j - \Delta t]] + E_C[n(j)C[\tau_j - \Delta t]] && (8)\\
&= E_C[C[\tau_j]C[\tau_j - \Delta t]] + \tilde{n}(j) && (9)\\
&= R_C[\Delta t] + \widetilde{n(j)} && (10)
\end{aligned}
$$

R_C is the auto-correlation function for the noise-free GPS spreading sequence and $\widetilde{n(j)}$ is white Gaussian noise, since multiplying Gaussian noise by a random zero-mean, unit variance sequence is still white Guassian noise with the same mean and variance. It follows that, using the Wiener Khinchin theorem:

$$S(f) = \mathscr{F}\left[R_C[\Delta t] + \tilde{n}(j)\right] = S_c(f)e^{-j2\pi f\Delta t} + \tilde{n}'(f) \tag{11}$$

where $\tilde{n}'(j)$ is still white Gaussian noise since the Fourier transform of time-domain independent Gaussian noise samples yields frequency-domain independent Gaussian noise samples. Due to the time-shifting property of the Fourier transform, Δt is simply the slope of the spectral phase.

$$\phi(f) = \tan^{-1}(S(f)) = 2\pi f\Delta t + \tilde{n}(f) \tag{12}$$

By dividing the one-sided precorrelation bandwidth Δf into N sub-intervals of width δf and center frequencies f_i where $i = 1, 2, ...N$, we can take measurements of $\phi(f_i)$ and use these measurements to estimate Δt by a weighted least squares fit to the slope of the measurements.

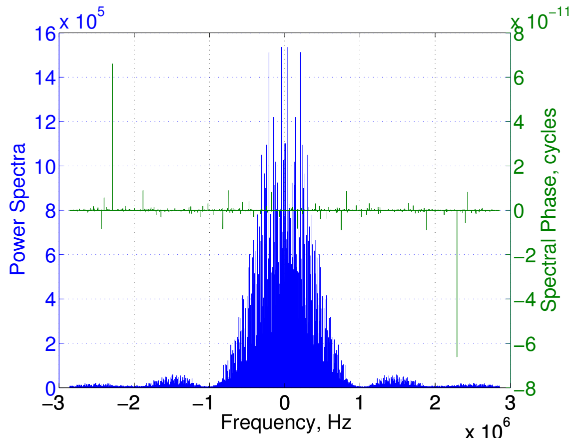

FIGURE 1. Plot showing power spectra and spectral phase for a $\Delta t = 0$ chips

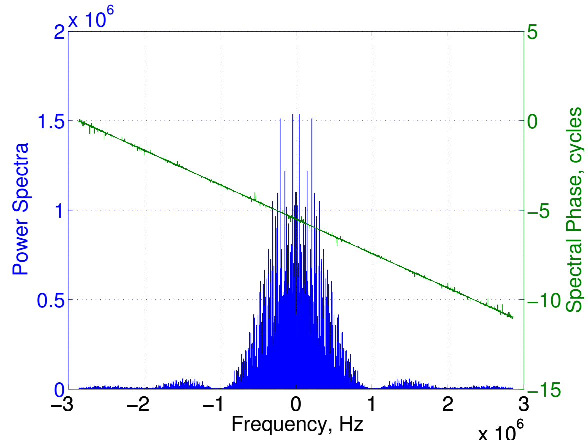

FIGURE 2. Plot showing power spectra and spectral phase for a $\Delta t = 2$ chips

Each phase measurement ϕ_i should weighted by its variance $\sigma^2_{\phi_i}$ which is a function of the average carrier-to-noise ratio in its particular frequency band. If carrier-to-noise ratio can be expressed as [1]:

$$\frac{C}{N_o} = \frac{1}{2\sigma^2_{IQ}T_a} \approx \frac{1}{2\sigma^2_{\phi}T_a}. \tag{13}$$

where T_a is the coherent integration time of the signal, and σ^2_{IQ} is the sample-variance, then

$$\sigma^2_{\phi_i} \approx \frac{1}{2\frac{C_i}{N_o}T_a} \tag{14}$$

where the effective signal power for the i^{th} measurement can be computed as

$$C_i = \int_{f_i-\frac{\delta f}{2}}^{f_i+\frac{\delta f}{2}} S_c(f)\mathrm{df} \approx \mathrm{S_c(f_i)df}. \tag{15}$$

We can stack these measurements up into a vector

$$\begin{bmatrix}\phi(f_1)\\ \phi(f_2)\\ \vdots\\ \phi(f_N)\end{bmatrix} = \underbrace{\begin{bmatrix}2\pi f_1\\ 2\pi f_2\\ \vdots\\ 2\pi f_N\end{bmatrix}}_{H}\left[\Delta t\right] + \begin{bmatrix}n_1\\ n_2\\ \vdots\\ n_N\end{bmatrix} \tag{16}$$

The measurement covariance matrix for a weighted least-squares problem can be given by:

$$R = \mathrm{diag}\left[\sigma^2_{\phi}(f_1), \sigma^2_{\phi}(f_2), ..., \sigma^2_{\phi}(f_N)\right] \tag{17}$$

The error variance of a weighted least squares estimate is defined as [2]

$$P = \left[H^T R^{-1} H\right]^{-1} \tag{18}$$

Thus, given the above setup, the error variance of the weighted-least-squares estimate of Δt can be easily derived as

$$\sigma^2_{\Delta t} = P = \left[H^T R^{-1} H\right]^{-1} \tag{19}$$

$$= \frac{1}{\sum_i (2\pi)^2 f_i^2 \frac{1}{\sigma^2_\phi(f_i)}} \tag{20}$$

$$= \frac{1}{\sum_i (2\pi)^2 f_i^2 \frac{1}{\frac{1}{2\frac{C_i}{N_o}T_a}}} \tag{21}$$

$$= \frac{1}{8\pi^2 \frac{1}{N_0} T_a \sum_i C_i f_i^2} \tag{22}$$

$$= \frac{1}{8\pi^2 \frac{1}{N_0} T_a \sum_i S_c(f_i) f_i^2 \mathrm{df}} \tag{23}$$

$$\tag{24}$$

and, in the limit as $N \to \infty$

$$= \frac{1}{8\pi^2 \frac{1}{N_0} T_a \int_{-\frac{\Delta f}{2}}^{\frac{\Delta f}{2}} S_c(f) f^2 df} \tag{25}$$

$$\tag{26}$$

Expanding $S_c(f)$ such that it is the product of the available signal power and the normalized power spectral density:

$$S_c(f) = C \cdot \tilde{S}_c(f) \tag{27}$$

where

$$\int_{-\infty}^{\infty} \tilde{S}_c(f) df = 1 \tag{28}$$

we can rewrite $\sigma^2_{\Delta t}$ as

$$\sigma^2_{\Delta t} = \frac{1}{8\pi^2 \frac{C}{N_0} T_a \int_{-\frac{\Delta f}{2}}^{\frac{\Delta f}{2}} \tilde{S}_c(f) f^2 \mathrm{df}} \tag{29}$$

$$\tag{30}$$

This expression for $\sigma^2_{\Delta t}$ is equivalent to the formal definition of the CRLB ([3], see appendix B). Thus the weighted least-squares estimator produces an estimate whose variance meets the CRLB.

References

[1] P. Misra and P. Enge, *Global Positioning System: Signals, Measurements, and Performance.* Lincoln, Massachusetts: Ganga-Jumana Press, revised second ed., 2012.

[2] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation.* New York: John Wiley and Sons, 2001.

[3] J. Betz and K. Kolodziejski, "Generalized theory of code tracking with an early-late discriminator part i: lower bound and coherent processing," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 45, no. 4, pp. 1538–1556, 2009.

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