Fault Free Integrity of Mid-Level Voting for Triplex Differential GPS Solutions

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BIOGRAPHIES

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ABSTRACT

Landing systems for large unmanned air vehicles have stringent integrity requirements as well as demanding system continuity requirements that often lead to triplex avionics architectures. Triplex avionics architectures are designs that have triple redundancy for key functions. Mid-level voting (MLV) algorithms that select the median value from among the three solutions are commonly used to select among available sensors and navigation solutions. When each solution is computed using a single suite of avionics, such median values are robust to single airborne sensor failures and provide improved unfaulted accuracy as well. Robustness to single faults results because a single faulted sensor will not impact the solutions computed by the other two sets of avionics. System accuracy is improved for zero mean error solutions because the median value is more concentrated about the truth than any of the single solutions. When performing fault tree analysis for integrity risk in the unfaulted case, it is common to treat sensors' errors as being mutually independent. In the case of multiple carrier phase differential GPS (CDGPS) solutions, this assumption is invalid due to common atmospheric errors and common reference receiver errors. This paper aims to quantify the unfaulted integrity risk from triplex correlated CDGPS solutions for float, fixed, and almost fixed baselines that use a MLV algorithm. The bound on the integrity risk is compared with that of independent solutions to show the impact of incorrectly assuming independence of CDGPS solutions. Triplex performance is compared to simplex to show improvement or degradation in unfaulted availability of integrity.

INTRODUCTION

The required navigation performance for new Carrier Phase Differential GPS (CDGPS) applications continues to become more demanding with subsequent generations. Ground based augmentation systems (GBAS) have strict requirements that the probability that the navigation system error exceeds an alert limit of 10 m without warning shall be less than an integrity risk on the order of 10^{-7} per approach. This leads to a relatively loose 95% accuracy requirement of 2 m [2]. Other navigation system applications, such as sea based landing and demonstration of autonomous aerial refueling, have meter-level ALs and decimeter-level accuracy requirements that necessitate carrier phase ambiguity resolution to provide sufficient accuracy [4, 6, 3]. The next generation of CDGPS use cases includes fully autonomous landing and refueling of large unmanned aerial vehicles (UAVs) in operational contexts. Operational use of unmanned systems will need even better continuity and integrity performance than previous demonstration programs. To satisfy these requirements, a variation of a triplex avionics architecture is likely to be used which has three complete sets of navigation equipment.

There are a number of estimation architectures to make use of the triplex equipment. Perhaps the simplest is a federated architecture in which each of the three avionics strings computes a separate CDGPS solution. The resulting solutions may be combined by averaging or by mid-level voting (MLV), which selects the median of the three values. MLV is often the preferred approach since it is more robust than averaging to single solution faults. But if any credit is to be taken for the integrity benefits of MLV, then the integrity risk associated with the algorithm must be bounded in all cases for which there is a non-negligible risk. For this, the joint distribution of the underlying variables must be taken into account, including any correlation among the solutions. If correlations among the solutions are neglected, then there will be a significant increase in the integrity risk of the overall solutions.

This paper develops methods to account for the integrity implications of MLV for CDGPS-based relative navigation systems. In the second section, expressions are obtained for MLV algorithm integrity risk for generally and jointly Gaussian distributed solutions. The third section applies the preceding theory to various types of CDGPS solutions including float solutions, fixed solutions, and almost fixed solutions. The fourth section provides an analytical comparison of MLV applied to federated triplex solutions with the same type of simplex (single string) solutions. The fifth section describes the simulation methodology used to evaluate algorithm performance. The sixth section shows simulation results, which demonstrate the performance improvement provided by the MLV algorithm when compared to simplex. The conclusion summarizes this work and looks to future related work.

MLV INTEGRITY FOR ARBITRARY DISTRIBUTIONS

A few preliminary definitions are needed before MLV can be evaluated. First, integrity risk is defined in this paper to be the probability that the error in the relative navigation solution currently in use exceeds a threshold, which is called an alert limit (AL), without a warning. A protection level (PL) is a value that is guaranteed to bound the error in the solution in use to a specified integrity risk. This paper, like those which precede it [6,4,8], uses protection levels to provide an *a priori* bound on the errors in a CDGPS solution based upon models of the system errors which must be validated to bound the errors in the actual system. Since these bounds are *a priori*, they are deterministic values for a given satellite geometry, hardware configuration, measurement set, and carrier phase track duration.

As such, the integrity risk is $IR = P\left(\varepsilon > AL \wedge \bar{W}\right)$. Where ε is the solution error, and $\bar{W}$ is the event that no warning is given. Protection levels are defined as a bound on solution error, $PL = \left\{x \mid P\left(\varepsilon > x\right) < IR_{spec}\right\}$, where IR_{spec} is the specified level of integrity risk. Any time the PL exceeds the AL, a warning is given. This implies an equivalent expression for integrity risk and a simplified bound if the PL is assumed to correctly bound the error:

$$\begin{aligned} IR &= P\left(\varepsilon > AL \wedge PL < AL\right) \\ &= P\left(\varepsilon > AL \mid PL < AL\right) P\left(PL < AL\right) \\ IR &\le \min \begin{Bmatrix} IR_{spec} \\ P\left(\varepsilon > AL \mid PL < AL\right) \end{Bmatrix} \end{aligned} \quad \textbf{(1)}$$

Since the unfaulted PL is deterministic when conditioned on a particular satellite geometry and measurement smoothing interval, the realization of the error is statistically independent of the PL value, yielding the final result:

$$IR \le \min \begin{Bmatrix} IR_{spec} \\ P\left(\varepsilon > AL\right) \end{Bmatrix} \quad \textbf{(2)}$$

The use of a MLV algorithm can be treated as conditioning on the knowledge of the order of the magnitudes of the three solutions [1]. That is:

$$\begin{aligned} &MLV \Rightarrow X_{(1)} \le X_{(2)} \le X_{(3)} \\ &X_{(1)}, X_{(2)}, X_{(3)} : \text{The sorted values} \end{aligned} \quad \textbf{(3)}$$

It is not known *a priori* which solution will be chosen by MLV, so in terms of the underlying random variables rather than the order statistic, we have:

$$\begin{aligned} &MLV \Rightarrow X_i \le X_j \le X_k \\ &\text{for some permutation } \{i, j, k\} \text{ of } \{1, 2, 3\} \end{aligned} \quad \textbf{(4)}$$

Based upon the preceding definitions, the *a priori* integrity risk represented by the chosen MLV solution is derived by first expressing the MLV condition in terms of the original random variables, second by the probability of the union of a set of events, and finally by recognizing that for a set of three solutions, the following simplification can be made:

$$\begin{aligned} &\left(\tilde{X}_i, \tilde{X}_j \ge AL\right) \cap \left(\tilde{X}_j, \tilde{X}_k \ge AL\right) \\ &\quad = \left(\tilde{X}_i, \tilde{X}_j, \tilde{X}_k \ge AL\right) \\ &\tilde{X}_{(2)} \triangleq X_{(2)} - x \\ &R_{MLV} = P\left(\left|\tilde{X}_{(2)}\right| \ge AL\right) \end{aligned}$$

$$R_{MLV} = P\left(\tilde{X}_{(2)} \geq AL\right) + P\left(\tilde{X}_{(2)} \leq -AL\right)$$

$$R_{MLV} = P\left(\bigcup_{j=1}^{3}\left[\left(\tilde{X}_{i_1}, \tilde{X}_{i_2 \neq i_1 \neq j} \geq AL\right)\right]\right) + P\left(\bigcup_{j=1}^{3}\left[\left(\tilde{X}_{i_1}, \tilde{X}_{i_2 \neq i_1 \neq j} \leq -AL\right)\right]\right)$$

$$R_{MLV} = \sum_{j=1}^{3}\left[P\left(\tilde{X}_{i_1}, \tilde{X}_{i_2 \neq i_1 \neq j} \geq AL\right)\right] - 2P\left(\tilde{X}_i \geq \underline{AL}, \forall i \in \{1,2,3\}\right) + \sum_{j=1}^{3}\left[P\left(\tilde{X}_{i_1}, \tilde{X}_{i_2 \neq i_1 \neq j} \leq -AL\right)\right] - 2P\left(\tilde{X}_i \leq \underline{AL}, \forall i \in \{1,2,3\}\right) \quad \textbf{(5)}$$

This is a general result that is independent of the distributions of the underlying random variables. In the case that the joint distribution of the X_i is multivariate normal, each of the event probabilities corresponds to an evaluation of the multivariate normal cumulative density function (CDF):

$$R_{MLV} = \sum_{j}\iiint_{V_j} f_{\underline{X}}\left(\underline{x}\right)\mathrm{d}\underline{x} - 2\sum_{k}\iiint_{V_k} f_{\underline{X}}\left(\underline{x}\right)\mathrm{d}\underline{x} \quad \textbf{(6)}$$

With V_j the regions defined by the individual events, and V_k, the regions defined by the intersections.

In the case that the underlying random variables are iid zero-mean normal, the above risk is greatly simplified:

$$R_{MLV} = \frac{3}{2}R_{simplex}^{2}\left(1 - \frac{R_{Simplex}}{3}\right) \quad \textbf{(7)}$$

It is important to note that the actual integrity risk will vary as a function of the degree of correlation among the three solutions between the values assuming complete independence and complete dependence. In the case of perfect correlation, the MLV integrity risk is equal to the simplex integrity risk. This illustrates the critical importance of correctly accounting for the correlations among solutions when computing MLV integrity risk.

Correlation Agnostic Integrity Risk Bound

In the case that there is insufficient knowledge of the correlations to accurately model the joint distribution of the three solutions, a simple *a priori* bound can be computed for the selected solution based upon the MLV criteria and the simplex integrity risks of the individual solutions:

$$R_{MLV} = P\left(\left|\tilde{X}_{(2)}\right| > AL\right)$$
$$\tilde{X}_{(1)} \leq \tilde{X}_{(2)} \leq \tilde{X}_{(3)} \Rightarrow \quad \textbf{(8)}$$
$$R_{MLV} \leq 1 - P\left(\left|\tilde{X}_{(1)}\right| < AL \,\&\, \left|\tilde{X}_{(3)}\right| < AL\right)$$

Again, since there is no way to no beforehand which solution will be chosen, the integrity risk bounds must be computed separately for each solution. Taking *i* to be the solution the integrity risk of which is to be bounded and *j* and *k* to be the other two solutions, the integrity bound for solution *i* is as follows:

$$R_i \leq 1 - \int_{-AL}^{AL}\int_{-AL}^{AL} f_{\tilde{X}_j,\tilde{X}_k}\left(\xi,\zeta\right)d\xi d\zeta \quad \textbf{(9)}$$

By first factoring the joint distribution via conditional probability, second recognizing the integrity risk of the individual solution, third recombining the joint distribution, and fourth bounding the remaining risk an upper bound is derived in equation 10.

$$\begin{aligned}
R_i &\leq 1 - \int_{-AL}^{AL} f_{\tilde{X}_j}\left(\xi\right)\int_{-AL}^{AL} f_{\tilde{X}_k|\tilde{X}_j}\left(\zeta \mid \tilde{X}_j = \xi\right)d\zeta d\xi \\
R_i &\leq 1 - \int_{-AL}^{AL} f_{\tilde{X}_j}\left(\xi\right)\cdot\left(1 - P\left(\left|\tilde{X}_k\right| > AL \mid \tilde{X}_j = \xi\right)\right)d\xi \\
R_i &\leq 1 - \int_{-AL}^{AL} f_{\tilde{X}_j}\left(\xi\right)d\xi + \int_{-AL}^{AL} f_{\tilde{X}_j}\left(\xi\right)\cdot P\left(\left|\tilde{X}_k\right| > AL \mid \tilde{X}_j = \xi\right)d\xi \\
R_i &\leq R_j + \int_{-AL}^{AL}\left\{f_{\tilde{X}_j}\left(\xi\right)\times\left[\int_{-\infty}^{-AL} f_{\tilde{X}_k|\tilde{X}_j}\left(\zeta \mid \tilde{X}_j = \xi\right)d\zeta + \int_{AL}^{\infty} f_{\tilde{X}_k|\tilde{X}_j}\left(\zeta \mid \tilde{X}_j = \xi\right)d\zeta\right]\right\}d\xi \\
R_i &\leq R_j + \int_{-\infty}^{-AL}\int_{-AL}^{AL} f_{\tilde{X}_j,\tilde{X}_k}\left(\xi,\zeta\right)d\xi d\zeta + \int_{AL}^{\infty}\int_{-AL}^{AL} f_{\tilde{X}_j,\tilde{X}_k}\left(\xi,\zeta\right)d\xi d\zeta \\
R_i &\leq R_j + \int_{-\infty}^{-AL}\int_{-\infty}^{\infty} f_{\tilde{X}_j,\tilde{X}_k}\left(\xi,\zeta\right)d\xi d\zeta + \int_{AL}^{\infty}\int_{-\infty}^{\infty} f_{\tilde{X}_j,\tilde{X}_k}\left(\xi,\zeta\right)d\xi d\zeta \\
R_i &\leq R_j + \int_{-\infty}^{-AL} f_{\tilde{X}_k}\left(\zeta\right)d\zeta + \int_{AL}^{\infty} f_{\tilde{X}_k}\left(\zeta\right)d\zeta \\
R_i &\leq R_j + R_k
\end{aligned} \quad \textbf{(10)}$$

This demonstrates that even if a single one of the solutions has very poor integrity, it can be assured that the integrity risk from the selected MLV solution can be limited to the sum of the integrity risks of the other two solutions. This is of no use when all three solutions are of similar quality, so it provides little benefit under nominal circumstances for fault free integrity. The bound can, however, be used to protect the integrity of the selected MLV solution in the presence of a single latent fault or single large integrity risk.

APPLICATION OF MLV TO CDGPS SOLUTIONS

To apply the analytical tools developed so far to triplex CDGPS solutions, the joint distributions of the three solutions must be derived for all types of CDGPS solutions of interest. This paper will examine the so called float solution, a fixed solution, and the Geometric Extra-Redundant Almost Fixed Solution (GERAFS) solution [4]. The Enforced Position-domain Integrity-risk of Cycle resolution (EPIC) algorithm [6] is not evaluated. As will be shown, it is not appropriate to consider the joint probabilities of ambiguity resolution among the federated solutions. The EPIC algorithm will be considered in a future paper on integrated triplex architectures where there is only a single primary CDGPS solution.

Triplex Float Solution Joint Distribution

Each individual float solution is formed by solving the linearized, least squares, double-difference relative baseline solution. The solution is formed by linearizing the measurement model about an initial estimate of the baseline vector. Assume that the measurements comprise a set carrier phase observables and some prior information, such as pseudoranges or geometry free estimates of the integer ambiguities. The joint covariance of the measurements and the prior information must be known to form a weighted least squares solution.

For the case that the prior information is a set of smoothed pseudoranges, when linearized about an initial estimate, $\bar{\underline{b}}$, the measurement model is as follows[4]:

$$\underline{y} = \underbrace{\begin{bmatrix} \mathbf{G} & \mathbf{\Lambda} \\ \mathbf{G} & 0 \end{bmatrix}}_{\mathbf{H}} \underline{x} + \underline{\varepsilon}, \text{with } \underline{\varepsilon} \sim \mathcal{N}\left(\underline{0}, \Sigma\right)$$

$$\Sigma : \text{Measurement covariance matrix}$$

$$\mathbf{G} = \begin{bmatrix} \left(\underline{u}_j^1 - \underline{u}_j^l\right)^T \\ \vdots \\ \left(\underline{u}_j^N - \underline{u}_j^l\right)^T \end{bmatrix}$$

$$\underline{u}_j^k : \text{unit vector from Rx } j \text{ to SV } k \tag{11}$$

$$\mathbf{\Lambda} = Diag\left(\lambda_1 \ldots \lambda_N\right) : \text{Wavelength matrix}$$

The associated float solution is computed and the solution matrix, **S**, is retained for later use:

$$\hat{\underline{x}} = \begin{bmatrix} \delta\hat{\underline{b}} \\ \hat{\underline{N}} \end{bmatrix} = \left(\mathbf{H}^T \Sigma^{-1} \mathbf{H}\right)^{-1} \mathbf{H}^T \Sigma^{-1} \underline{y} = \mathbf{S}\underline{y} \tag{12}$$

$$\underline{b} = \bar{\underline{b}} + \delta\underline{b} : \text{Final baseline estimate}$$

Each of these float solution matrices is combined with the total system joint measurement covariance to form the total solution covariance:

$$\begin{aligned} \Sigma_{float,triplex} &= Cov\left(\left[\hat{\underline{b}}_1 \ \hat{\underline{N}}_1 \middle| \hat{\underline{b}}_2 \ \hat{\underline{N}}_2 \middle| \hat{\underline{b}}_3 \ \hat{\underline{N}}_3\right]^T\right) \\ &= \begin{bmatrix} \mathbf{S}_1 & 0 & 0 \\ 0 & \mathbf{S}_2 & 0 \\ 0 & 0 & \mathbf{S}_3 \end{bmatrix} \begin{bmatrix} \Sigma_{y,1} & \Sigma_{y,2,1}^T & \Sigma_{y,3,1}^T \\ \Sigma_{y,2,1} & \Sigma_{y,2} & \Sigma_{y,3,2}^T \\ \Sigma_{y,3,1} & \Sigma_{y,3,2} & \Sigma_{y,3} \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & 0 & 0 \\ 0 & \mathbf{S}_2 & 0 \\ 0 & 0 & \mathbf{S}_3 \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{S}_1 \Sigma_{y,1} \mathbf{S}_1^T & \mathbf{S}_1 \Sigma_{y,2,1}^T \mathbf{S}_2^T & \mathbf{S}_1 \Sigma_{y,3,1}^T \mathbf{S}_3^T \\ \mathbf{S}_2 \Sigma_{y,2,1} \mathbf{S}_1^T & \mathbf{S}_2 \Sigma_{y,2} \mathbf{S}_2^T & \mathbf{S}_2 \Sigma_{y,3,2}^T \mathbf{S}_3^T \\ \mathbf{S}_3 \Sigma_{y,3,1} \mathbf{S}_1^T & \mathbf{S}_3 \Sigma_{y,3,2} \mathbf{S}_2^T & \mathbf{S}_3 \Sigma_{y,3} \mathbf{S}_3^T \end{bmatrix} \end{aligned} \tag{13}$$

The off-block-diagonal terms of the measurement covariance matrix are populated by reference receiver errors, atmospheric errors, and lever arm errors used to translated the three solutions to a common reference point. The resulting triplex covariance can be broken down into several pieces: covariance among the various baseline estimates, among the various real valued float integer estimates, and between the baseline estimates and the integer estimates. This decomposition of the matrix is denoted as follows, with the upper triangular portion being the transpose of the lower triangular portion:

$$\left[\begin{array}{cc|cc|cc} \Sigma_{b_1} & & & & & \\ \Sigma_{N_1 b_1} & \Sigma_{N_1} & & & & \\ \hline \Sigma_{b_2 b_1} & \Sigma_{b_2 N_1} & \Sigma_{b_2} & & & \\ \Sigma_{N_2 b_1} & \Sigma_{N_2 N_1} & \Sigma_{N_2 b_2} & \Sigma_{N_2} & & \\ \hline \Sigma_{b_3 b_1} & \Sigma_{b_3 N_1} & \Sigma_{b_3 b_2} & \Sigma_{b_3 N_2} & \Sigma_{b_3} & \\ \Sigma_{N_3 b_1} & \Sigma_{N_3 N_1} & \Sigma_{N_3 b_2} & \Sigma_{N_3 N_2} & \Sigma_{N_3 b_3} & \Sigma_{N_3} \end{array}\right] \tag{14}$$

By extracting the desired values from this overall joint solution covariance matrix, MLV can be performed on the vertical and lateral components of the relative baseline solution.

Simplex Fixed Solution Integrity

Continuing on from the float solution obtained in the previous section, there exist many methods to attempt to fix the integer ambiguities. Among these are integer rounding [7], integer bootstrap [5], and Least-squares AMBiguity Decorrelation Adjustment (LAMBDA) [8]. Each algorithm has its advantages and disadvantages.

Integer rounding is the simplest, but it has the lowest probability of correctly fixing the integers. Bootstrap has improved probability of success and has a convenient way to predict probability of correct fix (P_{CF}), but it is sub-optimal and is sensitive to the order and combination in which ambiguities are resolved. LAMBDA is an optimal method in terms of P_{CF}, but entails a least-squares search of the integer space and has no simple method to predict P_{CF}. High integrity CDGPS systems typically use the bootstrap method together with the ambiguity decorrelation adjustment of the LAMBDA method. By using the decorrelated ambiguities, the bootstrap algorithm fixes successive integer ambiguities in the order of maximum conditional P_{CF}. This set of algorithms provides high integrity with predictable probability of correct integer fixing.

The fixed baseline solution is not distributed according to a Gaussian distribution. Rather, it is a mixture of multivariate Gaussian distributions of equal covariance, but differing means [5]:

$$\begin{aligned} &\breve{\underline{N}}_i : \text{Fixed integer ambiguity estimate} \\ &\breve{\underline{b}}_i = \hat{\underline{b}} - \Sigma_{bN}\Sigma_N^{-1}\left(\hat{\underline{N}} - \breve{\underline{N}}_i\right) \\ &\Sigma_{\breve{b}} = \Sigma_b - \Sigma_{bN}\Sigma_N^{-1}\Sigma_{Nb} \\ &\underline{b} \sim \sum_{\breve{\underline{N}}_i \in \mathbb{Z}^n} \mathcal{N}\left(\breve{\underline{b}}_i, \Sigma_{\breve{b}}\right), \text{w.p. } P\left(N = \breve{\underline{N}}_i\right) \end{aligned} \tag{15}$$

Assuming that the fixed solution has found the correct set of integer ambiguities, the baseline estimate is once again a zero-mean multivariate Gaussian random vector. The integrity risk in making this assumption is P_{CF}. The covariance of the solution is now much smaller than the float covariance since the errors are driven by the carrier phase measurements which are of significantly higher quality than the prior information from pseudoranges. The integrity risk associated with the vertical component of a fixed solution is bounded by the following:

$$\begin{aligned} R_{fixed,V} &= P\left(\left|\underline{b}_V - \underline{b}_V\right| > AL_V\right) \\ &\le 1 - P_{CF} + P_{CF} P\left(\left|\underline{b}_V - \underline{b}_V\right| > AL_V \mid N = \breve{\underline{N}}\right) \\ &\le 1 - P_{CF} + P_{CF}\left[1 - erf\left(\frac{AL_V}{\sqrt{2}\sigma_{fixed,V}}\right)\right] \\ &\le 1 - P_{CF}\, erf\left(\frac{AL_V}{\sqrt{2}\sigma_{fixed,V}}\right) \end{aligned} \tag{16}$$

Triplex Fixed Solution Joint Distribution

The transition from simplex to triplex for the float solution is simply a matter of extracting the appropriate portions of the system float covariance matrix to form the baseline covariance, but the fixed triplex solution is more complicated. Consider the fixed baseline covariance:

$$\Sigma_{\breve{\underline{B}},triplex} = \begin{bmatrix} \Sigma_{b_1|N_1} & & \\ \Sigma_{b_2b_1|N_2N_1} & \Sigma_{b_2|N_2} & \\ \Sigma_{b_3b_1|N_3N_1} & \Sigma_{b_3b_2|N_3N_2} & \Sigma_{b_3|N_3} \end{bmatrix} \tag{17}$$

Where the off diagonal terms are computed from parts of the system float covariance and the block diagonal terms are the results of the individual fixed solutions:

$$\begin{aligned} \Sigma_{b_jb_i|N_jN_i} &= \Sigma_{b_jb_i} \\ &- \begin{bmatrix} \Sigma_{N_ib_i} \\ \Sigma_{N_jb_i} \end{bmatrix}^T \begin{bmatrix} \Sigma_{N_i} & \Sigma_{N_iN_j} \\ \Sigma_{N_jN_i} & \Sigma_{N_j} \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{N_ib_i} \\ \Sigma_{N_jb_i} \end{bmatrix} \end{aligned} \tag{18}$$

The integrity of the MLV algorithm for the triplex fixed solutions depends upon all three of the fixes being correct. Conditioned on this event, the three fixed baselines are distributed as a single, correlated, zero-mean multivariate Gaussian random vector:

$$\begin{aligned} R_{fixed,triplex} &= 1 - P\left(CF_1 \cap CF_2 \cap CF_3\right) \\ &+ P\left(CF_1 \cap CF_2 \cap CF_3\right) \\ &\times R_{MLV}\left(AL, \underline{0}, \Sigma_{\breve{\underline{B}},triplex}\right) \end{aligned} \tag{19}$$

To upper bound the integrity risk, the probability of all three integer fixed solutions must both upper and lower bounded since it appears as both a positive and negative quantity. The upper bound is attained by assuming that the events are perfectly correlated, i.e. one correct fix implies that all others are correctly fixed and one incorrect fix implies all others are incorrectly fixed. The lower bound is obtained by making the opposite assumption, that each fix is statistically independent. The reality is something between the two since the measurements used in the solutions are only partially correlated:

$$\begin{aligned} &P\left(\bigcap_i CF_i\right) = \\ &\quad P\left(CF_{i_1} \mid CF_{i_2}, CF_{i_3}\right) P\left(CF_{i_2} \mid CF_{i_3}\right) P\left(CF_{i_3}\right) \\ &\text{Perfect correlation} \Rightarrow \\ &\quad P\left(CF_{i_2} \mid CF_{i_3}\right) = 1, \forall i_2, i_3 \\ &\therefore P\left(\bigcap_i CF_i\right) \le \min_i\left(P_{CF_i}\right) \end{aligned}$$

$$\text{Independence} \Rightarrow$$

$$P\left(CF_{i_2} \mid CF_{i_3}\right) = P\left(CF_{i_2}\right), \forall i_2, i_3 \quad \textbf{(20)}$$

$$\therefore \prod_i P_{CF_i} \le P\left(\bigcap_i CF_i\right) \le \min_i\left(P_{CF_i}\right)$$

Apart from the detailed information that would be obtained from processing a fully integrated triplex fixed solution, these bounds are the closest that can be obtained for the probability of correct fix. Substituting these bounds into the integrity equation yields the bound on triplex float integrity risk:

$$R_{fixed,MLV} \le 1 - \prod_i P_{CF_i} + \min_i\left(P_{CF_i}\right) \times R_{MLV}\left(AL, \underline{0}, \Sigma_{\breve{\underline{B}},triplex}\right)$$

The corresponding protection level is given by iteration of the integrity risk equation for varying levels of error until the integrity risk approaches the specified level of risk.

It is evident from examining the above expression that this solution is only available if the product of the probabilities of correct fix is very close to one. In fact, the demands on the probability of correct fix become even more stringent than for the simplex case so that the assumption that all three fixes are correct can hold. This may cause the overall system availability to be even lower for a federated triplex fixed solution than for the simplex solution if the P_{CF} is near the required integrity risk.

Simplex GERAFS Integrity Risk

If the fixed solutions do not produce a sufficiently high probability of correct fix, then an alternative is to use an "almost fixing" solution. Almost fixing solutions control integrity risk by assessing the risk induced by biases which result from incorrect fixes near the fixed solution. Two such algorithms are GERAFS [4] and EPIC [6].

In the almost fixed case, the fixed baseline has the same underlying multi-modal mixture distribution, but more care is taken to address the impact of modes other than the chosen fixed solution. Both algorithms examine a set of candidate integer fixes, $\breve{\underline{N}}$, which correspond to the fixed solution plus integer error vectors. The set of candidates considered by the algorithm is called the enlarged pull in region (EPIR). As described, each candidate fix shifts the solution by a deterministic bias.

The GERAFS algorithm addresses the integrity risk by accounting for the worst case bias induced by any candidate fix and neglecting the subtler points of the distribution. The magnitude of the worst case bias is called the incorrect fixing bias (*IFB*):

$$P_{NAF} = 1 - \left(P_{CF} + P_{EPIR}\right) = 1 - P_{AF}$$

$$\begin{aligned} R_{GERAFS,V} &= P\left(\left|\breve{\underline{b}}_V - \underline{b}_V\right| > AL_V\right) \\ &\le P_{NAF} + P_{CF} P\left(\left|\breve{\underline{b}}_V - \underline{b}_V\right| > AL_V \mid N = \breve{\underline{N}}\right) \\ &\quad + P_{EPIR} P\left(\left|\breve{\underline{b}}_V - \underline{b}_V\right| > AL_V \mid N \in EPIR\right) \end{aligned}$$

$$R_{GERAFS,V} \le P_{NAF} + P_{CF}\, erf\left(\frac{AL_V}{\sqrt{2}\sigma_{fixed,V}}\right) + P_{EPIR}\left(1 - \frac{1}{2}\left[\begin{aligned} & erf\left(\frac{AL_V + IFB}{\sqrt{2}\sigma_{fixed,V}}\right) \\ & + erf\left(\frac{-AL_V - IFB}{\sqrt{2}\sigma_{fixed,V}}\right)\end{aligned}\right]\right) \quad \textbf{(21)}$$

Triplex GERAFS Joint Distribution

The covariance of the almost fixed solutions are the same as that of the fixed solutions, so all that remains is to assess the integrity ramifications of almost fixing. The development is similar to the fixed case with P_{AF} often taking the role of P_{CF} for the fixed case. The MLV integrity monitor must also be modified to reflect the worst case biases in the positive and negative directions:

$$\begin{aligned} R_{MLV,GERAFS} &= \sum_j \iiint_{V_j^+} f_{\underline{X}}\left(\underline{x} \mid \mu = IFB\right) d\underline{x} \\ &\quad - 2\iiint_{V_k^+} f_{\underline{X}}\left(\underline{x} \mid \mu = IFB\right) d\underline{x} \\ &\quad + \sum_j \iiint_{V_j^-} f_{\underline{X}}\left(\underline{x} \mid \mu = -IFB\right) d\underline{x} \\ &\quad - 2\iiint_{V_k^-} f_{\underline{X}}\left(\underline{x} \mid \mu = -IFB\right) d\underline{x} \end{aligned} \quad \textbf{(22)}$$

$$V_j^+ \in \left\{\begin{aligned} &\{[-\infty, AL, AL], [\infty, \infty, \infty]\}, \\ &\{[AL, -\infty, AL], [\infty, \infty, \infty]\}, \\ &\{[AL, AL, -\infty], [\infty, \infty, \infty]\} \end{aligned}\right\},$$

With,

$$V_j^- \in \left\{\begin{aligned} &\{-[\infty, \infty, \infty], -[\infty, AL, AL]\}, \\ &\{-[\infty, \infty, \infty], -[AL, \infty, AL]\}, \\ &\{-[\infty, \infty, \infty], -[AL, AL, \infty]\} \end{aligned}\right\}$$

$$V_k^+ = \{[AL, AL, AL], [\infty, \infty, \infty]\}$$

$$V_k^- = \{-[\infty, \infty, \infty], -[AL, AL, AL]\}$$

This is identical to the previous definition of MLV integrity except that the magnitudes of the means are applied in the direction that maximizes risk. That is, *IFB* is added when evaluated against positive errors and subtracted when evaluated against negative errors. This ensures conservatism in the integrity bound:

$$\begin{aligned} &P_{AF_i} = P_{CF_i} + P_{EPIR_i} \\ &R_{GERAFS,\mathrm{MLV},V} \le 1 - \prod_i P_{AF_i} \\ &\quad + \min_i \left(P_{CF_i}\right) \times R_{MLV}\left(AL, \underline{0}, \Sigma_{\breve{B}_V,triplex}\right) \\ &\quad + \begin{Bmatrix} \min_i \left(P_{EPIR_i}\right) \\ \times R_{MLV,GERAFS}\left(AL, IFB, \Sigma_{\breve{B}_V,triplex}\right) \end{Bmatrix} \end{aligned} \quad \mathbf{(23)}$$

ANALYTICAL COMPARISON TO SIMPLEX SOLUTIONS

The important metrics to be considered when comparing the simplex and triplex solutions are solution accuracy and solution availability of integrity. Availability of integrity is the percentage of time that a particular system can meet the specified integrity requirements in operation. The integrity risk equations for each type of solution provide the basis for making these assessments.

For the float solution, consider that the integrity risk is a relatively simple expression since the solution is formed by weighted least squares without applying the integer constraints to the ambiguity resolution. Because of this there is no automatic integrity penalty for assuming that the integer ambiguities have been correctly resolved. This implies that correctly accounting for the integrity risk of MLV for float solutions can only decrease when compared to the simplex solution. Since the MLV equations apply to any arbitrary AL, the AL can be substituted with a PL to evaluate arbitrary levels of risk. If the level of risk corresponds to a desired accuracy level, e.g. 95%, then the MLV equation demonstrates that MLV also improves the accuracy of federated triplex float solutions when compared to simplex float solutions. Depending on the particular values of integrity risk and alert limits, MLV may provide sufficient improvement to make a float solution viable when it otherwise would not be.

For the federated triplex fixed solution, MLV is not usually advantageous. Typically, the integrity risk for a fixed solution is limited by P_{CF}, not by the accuracy of the fixed solution. Examination of the integrity equation for MLV fixed solutions shows that all three solutions are required to have simultaneously correct fixes. This drives the individual required P_{CF} requirement to 1/3rd the original simplex requirement. If a fixed solution is unavailable, or marginally available for the simplex case, it is less available for virtually all conditions in the federated triplex case.

Federated triplex GERAFS solutions are not analytically clear cut. They have an analogous problem to the fixed solution in that the P_{NAF} requirement for each solution becomes more demanding, but the impact of this change is not binary loss of availability as in the fixed case. Rather consider that GERAFS accounts for the impact of incorrect fixes in the EPIR by using the worst case *IFB* to bound the integrity risk. In this case, as long as the new P_{NAF} requirement can be satisfied, the impact is absorbed in a possible increase in the IFB. Depending on the margin between the IFB and the AL, the larger IFB can be mitigated by a reduced tail probability from the application of MLV to the remaining solutions.

SIMULATION METHODOLOGY

A numerical covariance analysis tool was used to assess the performance of the federated triplex GERAFS solutions as compared to the simplex GERAFS solution. This tool is an updated version of the same availability model (AM) that was originally used to assess the availability of the GERAFS algorithm [4]. The model was updated to include the position solution and integrity bounds described in this paper and a more stringent integrity requirement to reflect the needs of unmanned air vehicles. All solutions are computed using the wide lane carrier phase and narrow lane code combination.

The primary metric used to compare the performance of the two algorithms is solution availability. A solution is defined to be available at a given place and time if it satisfies both accuracy and integrity requirements. Accuracy compliance is assessed as a daily average accuracy at a given location for a given satellite constellation. Integrity is assessed as available if instantaneous solution integrity risk less than required. The AM evaluates algorithm performance on a worldwide grid of latitude and longitude at 15 minute intervals over a 24 hour period.

Because integrity is an instantaneous requirement, there are 96*$N_{\mathrm{GridPoints}}$ assessments of availability of integrity. Alternatively, accuracy is a daily average for each location, so there are only $N_{\mathrm{GridPoints}}$ assessments of availability of accuracy. Each worldwide grid point is given an availability value defined as follows with world wide availability is computed as the average over all grid points:

$$A_{GP} = \begin{cases} \dfrac{\#\left\{t \mid IR_{GP}\left(t\right) < IR_{spec}\right\}}{\#\left\{t\right\}} & \varepsilon_{avg} \le \varepsilon_{spec} \\ 0 & \varepsilon_{avg} > \varepsilon_{spec} \end{cases} \quad \mathbf{(24)}$$

The error models used for the simulation include a thermal noise, multipath, and antenna bias model for carrier phase and pseudorange, lever arm translation errors, and propagation effects due to latency of reference receiver data. Both simplex and triplex GERAFS implementations use a float solution as a backup when the P_{CF} and P_{AF} requirements are not satisfied. This allows for graceful degradation of system average accuracy as the percentage of time that the GERAFS algorithm is unable to fix increases.

SIMULATION RESULTS

The results of the original GERAFS algorithm [] have been reproduced with an integrity risk requirement reduced to reflect the needs of an unmanned landing. These results plot the availability of the solutions as a function of the accuracy that they provide for varying ALs. There is a general trend for each solution that availability decreases with the AL. The GERAFS and Federated GERAFS solutions also have decreasing accuracy as the AL decreases as well. This trend is a result of the backup float solution having to be used when GERAFS is not able to satisfy the required integrity risk for the given AL.

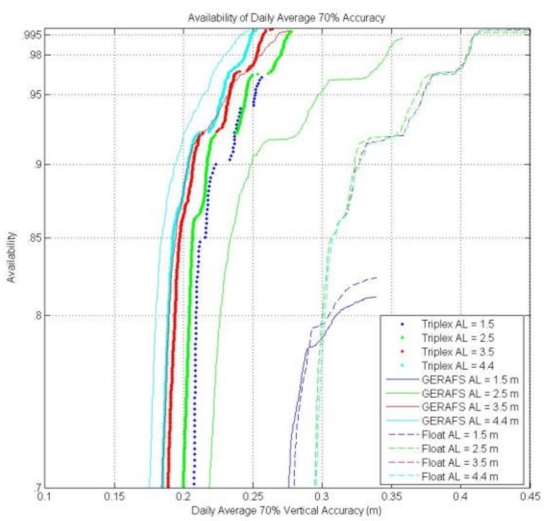

Figure 1: Worldwide availability of daily average accuracy for various solutions at varying alert limits

The accuracy of the GERAFS solution exhibits significant variation depending upon the AL that it must satisfy. For the least restrictive AL = 4.4 m, The GERAFS solution provides the best accuracy performance, because it is almost always able to operate in the almost fixed mode. However, as the AL is reduced, GERAFS is less able to satisfy the specified P_{AF}, which results in the use of the backup float solution. The direct results of the reduced AL are degraded average accuracy and reduced availability of integrity. For the specified integrity risk used in these simulations, GERAFS is usually unable to satisfy P_{AF} for an AL of 1.5 m. As a result, for this combination of requirements, simplex GERAFS performance is roughly equivalent to simplex float performance.

The triplex solution is often unable to satisfy the reduced P_{AF} required to simultaneously fix all three solutions. This can be inferred from the fact that the triplex solution is not significantly more accurate than the simplex GERAFS solution. For AL equal to either 3.5 m or 4.4 m, the triplex solution is still able to fix the integers occasionally, but if the AL is 2.5 m or less, the triplex solution is operating in float mode. Even though the triplex solution is less able to fix the integers than the simplex solution, the MLV algorithm provides enough improvement in the float solution accuracy that the triplex solution provides better availability than simplex GERAFS for each AL considered and better accuracy at the 99.5% availability level for all ALs evaluated except 4.4 m.

Table 1: World-wide availability for varying ALs for each solution

Solution \ AL	1.5 m	2.5 m	3.5 m	4.4 m
Float	.8239	.9977	.9999	.9999
GERAFS	.8117	.9929	.9984	.9990
Triplex	.9628	.9977	.9999	.9999

Table 2: Daily average 70% accuracy for each solution at 99.5% availability or maximum obtained

Solution \ AL	1.5 m	2.5 m	3.5 m	4.4 m
Float	33.8 cm	39.8 cm	39.9 cm	39.9 cm
GERAFS	34.7 cm	35.8 cm	26.8 cm	24.3 cm
Triplex	25.6 cm	27.5 cm	25.8 cm	24.9 cm

These results indicate that for this set of requirements, the overall system performance will be optimized by performing MLV when the solutions are floating but by not taking any credit for MLV in the triplex GERAFS case. This allows each GERAFS solution to use the full P_{AF} allocation and protect its own integrity while gaining the accuracy benefits of MLV.

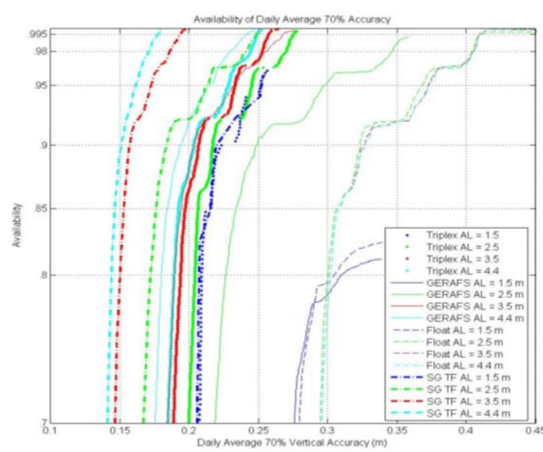

Figure 2: Alternative triplex implementation performance comparison

Utilizing MLV in this alternate manner improves accuracy performance by increasing the likelihood of using the GERAFS solution while gaining the accuracy and integrity benefits of MLV for a float solution. If the final integrity requirements are close to a 1.5 m AL, then further algorithmic improvements will be needed to increase P_{AF}. These may be obtained by using an integrated architecture, which will be analyzed in subsequent research.

CONCLUSION

Federated triplex solutions offer significant benefits to improve accuracy and availability of float solutions, but for likely levels of integrity risk and alert limits the additional burden of correctly fixing all three sets of integers prevents performance improvement for fixed or almost fixed solutions. The degree of improvement afforded by federated triplex float solutions makes them competitive with simplex GERAFS solutions. Unfortunately, this solution requires three active rover receivers at all times which would require even more receivers to be used to ensure system continuity. MLV for the GERAFS algorithm still provides improved accuracy even when no additional integrity credit is claimed. To alleviate the continuity risk and enhance integrity, future studies will examine integrated architecture alternatives which will provide performance improvements with fewer rover receivers.

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REFERENCES

[1] G. Casella and R. L. Berger, "Order Statistics," in *Statistical Inference*, Pacific Grove, CA, Duxbury, 2002, pp. 226-231.

[2] "Minimum Aviation System Performance Standards for the Local Area Augmentation System Airborne Equipment RTCA Document Number DO-245A," 2004.

[3] S. Khanafseh and B. Pervan, "Detection and Mitigation of Reference Receiver Faults in Differential Carrier Phase Navigation Systems," *IEEE TRANSACTIONS ON AEROSPACE AND ELECTRONIC SYSTEMS,* vol. 47, no. 4, pp. 2391-2404, 2011.

[4] S. Wu, S. R. Peck, R. M. Fries and G. A. McGraw, "Geometry Extra-Redundant Almost Fixed Solutions: A High Integrity Approach for Carrier Phase Ambiguity Resolution for High Accuracy Relative Navigation," *Position, Location and Navigation Symposium, 2008 IEEE/ION,* pp. 568-582, 2008.

[5] P. Teunissen, "Theory of Carrier Phase Ambiguity Resolution," *Wuhan University Journal of Natural Sciences,* vol. 8, no. 2B, pp. 471-484, 2003.

[6] S. Khanafseh and B. Pervan, "New Approach for Calculating Position Domain Integrity Risk for Cycle Resolution in Carrier Phase Navigation Systems," *Aerospace and Electronic Systems, IEEE Transactions on,* vol. 46, no. 1, pp. 296 - 307, January 2010.

[7] P. Teunissen, "Success Probability of Integer GPS Ambiguity Rounding and Bootstrapping," *Journal of Geodesy,* vol. 72, pp. 606-612, 1998.

[8] P. Teunissen, P. de Jong and C. Tiberius, "The least-squares ambiguity decorrelation adjustment: Its performance on short GPS baselines and short observation spans," *Journal of Geodesy,* vol. 71, no. 10, pp. 589-602, 1997.